Problem 3

Suppose a metal rod loses heat across the lateral boundary according to the equation

$$u_t = \alpha^2 u_{xx} - \beta u \qquad 0 < x < 1$$

and suppose we keep the ends of the rod at $u(0,t) = 1$ and $u(1,t) = 1$. Find the steady-state temperature of the rod (graph it). Where is heat flowing in this problem?

Solution

Take advantage of the fact that the heat equation is linear: Assume the solution for the temperature $u(x,t)$ has a steady component and a transient component.

$$u(x,t) = R(x) + U(x,t)$$

Substitute this formula into the PDE.

$$\frac{\partial}{\partial t}[R(x) + U(x,t)] = \alpha^2 \frac{\partial^2}{\partial x^2}[R(x) + U(x,t)] - \beta[R(x) + U(x,t)]$$

Evaluate the derivatives.

$$U_t = \alpha^2[R''(x) + U_{xx}] - \beta[R(x) + U(x,t)]$$

If we set

$$\alpha^2 R''(x) - \beta R(x) = 0,$$

then the previous equation becomes

$$U_t = \alpha^2 U_{xx} - \beta U.$$

Now substitute the formula for $u(x,t)$ into the boundary conditions.

$$\begin{aligned} u(0,t) &= 1 \qquad \rightarrow \qquad R(0) + U(0,t) = 1 \\ u(1,t) &= 1 \qquad \rightarrow \qquad R(1) + U(1,t) = 1 \end{aligned}$$

Set $R(0) = 1$ and $R(1) = 1$ so that $U(0,t) = 0$ and $U(1,t) = 0$. Solve the ODE for the steady-state temperature.

$$R''(x) = \frac{\beta}{\alpha^2} R(x)$$

Since the rod is losing heat along its lateral sides, $\beta > 0$. That means the general solution for $R(x)$ can be written in terms of hyperbolic sine and hyperbolic cosine.

$$R(x) = C_1 \cosh\left(\frac{\sqrt{\beta}}{\alpha}x\right) + C_2 \sinh\left(\frac{\sqrt{\beta}}{\alpha}x\right)$$

Apply the two boundary conditions to determine C_1 and C_2.

$$\begin{aligned} R(0) &= C_1 = 1 \\ R(1) &= C_1 \cosh\left(\frac{\sqrt{\beta}}{\alpha}\right) + C_2 \sinh\left(\frac{\sqrt{\beta}}{\alpha}\right) = 1 \end{aligned}$$

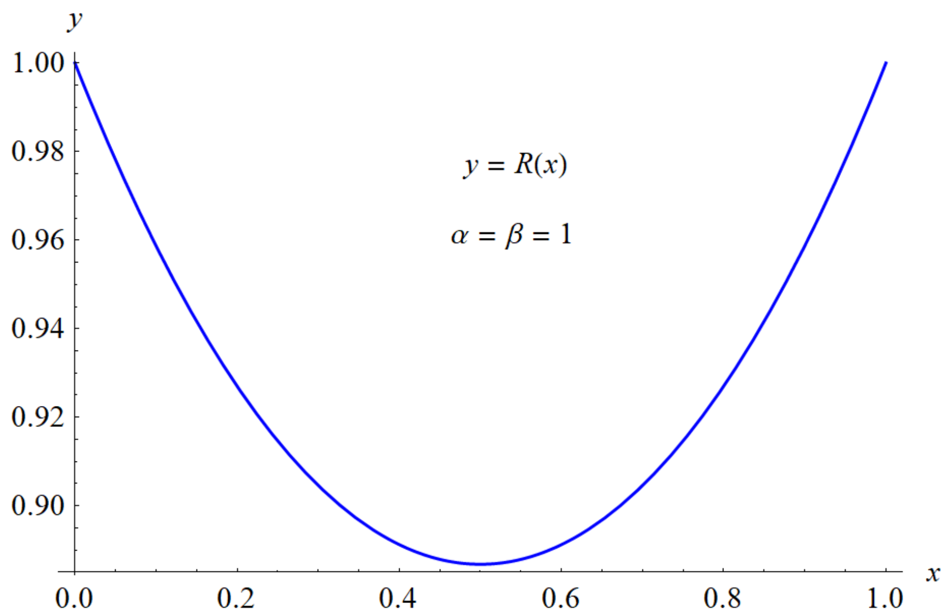
Solving this system yields

$$C_1 = 1 \quad \text{and} \quad C_2 = \frac{1 - \cosh\left(\frac{\sqrt{\beta}}{\alpha}\right)}{\sinh\left(\frac{\sqrt{\beta}}{\alpha}\right)}.$$

Therefore, the steady-state temperature is

$$R(x) = \cosh\left(\frac{\sqrt{\beta}}{\alpha}x\right) + \frac{1 - \cosh\left(\frac{\sqrt{\beta}}{\alpha}\right)}{\sinh\left(\frac{\sqrt{\beta}}{\alpha}\right)} \sinh\left(\frac{\sqrt{\beta}}{\alpha}x\right).$$

To get an idea what this looks like, graph $R(x)$ versus x with $\alpha = 1$ and $\beta = 1$.



According to Fourier's law of heat conduction, the heat flux is given by

$$q = -\alpha^2 \frac{\partial u}{\partial x}.$$

Wherever the slope of the temperature curve is negative, the heat flows in the positive x-direction. And wherever the slope of the temperature is positive, the heat flows in the negative x-direction. In the graph above, heat flows in the positive x-direction for $0 < x < 0.5$ and flows in the negative x-direction for $0.5 < x < 1$.